Solutions for Final Exam

MAS501 Analysis for Engineers, Spring 2011

---

1. (a) False. See Problem 6.1.3 on the Ash's book, page 97.

   (b) True. Done in the class; use Theorem 6.1.2 and Theorem 6.4.1 to prove it.

   (c) False. If f' is continuous and bounded on $(0,1)$, then f is also bounded by the Fundamental Theorem of Calculus.

   (d) True.

   (e) True. Using integration by parts, we have

   $$\int_a^b x f(x) f'(x)\, dx = \frac{1}{2}\int_a^b x(f^2(x))'\, dx = -\frac{1}{2}\int_a^b f^2(x)\, dx = -\frac{1}{2} < 0.$$

   (f) True. Because the integrand is an odd function, $\int_{-1}^2 x\sqrt{1+x^2}\, dx = \int_1^2 x\sqrt{1+x^2}\, dx$. Now we can make the change of variable $y = 1 + x^2$ to conclude

   $$\int_1^2 x\sqrt{1+x^2}\, dx = \int_2^5 \frac{1}{2}\sqrt{y}\, dy = \frac{1}{3}(5^{3/2} - 2^{3/2}).$$

   Finally note that $5^{3/2} > 5$ and $2^{3/2} < 2^2 = 4$ to complete the proof.

   (g) False. By the Mean Value Theorem for Integrals, we have $\lim_{h\to 0}\int_{a-h}^{a+h} f(x)\, dx = 2f(a)$.

   (h) True. See Theorem 7.2.1 on the Ash's book, page 122.

   (i) True. It follows from Weierstrass M-Test and Theorem 7.2.1 on the Ash's book, page 122.

   (j) False. There is a bounded, continuous but nowhere differentiable function on **R**. See section 7.3.3 on the Ash's book.

   (k) True. The proof is contained in the answer of Problem 7.4.3 on the Ash's book, page 133.

   (l) False. Every countable set has outer measure zero and every set of outer measure zero is measurable.

   (m) False. See Problem 3.18 on the Royden's book, page 70.

   (n) True. Every continuous function is measurable. And the limit supremum of measurable sets is also measurable.

   (o) True. This is Fatou's Lemma.

2. *Proof.* First note that when $x = 0$, the inequality holds for every real number a. Now fix a positive real number x. Then, by Taylor's formular with remainder, we have

   $$\ln(1+x) = x - \frac{1}{2\,(1+y)^2}\, x^2 \quad \text{for some } y \in (0, x).$$

   Because $0 < y < x$, it holds that

   $$ax^2 \geq x - \ln(1+x) = \frac{1}{2\,(1+y)^2}\, x^2 \geq \frac{1}{2\,(1+x)^2}\, x^2$$

or

$$a \geq \frac{1}{2\,(1+x)^2}.$$

Since $x > 0$ is arbitrary, we have $a \geq 1/2$. On the other hand, for $y > 0$, we have

$$x - \ln(1+x) = \frac{1}{2\,(1+y)^2}\,x^2 \leq \frac{1}{2}\,x^2.$$

Therefore the answer is $a = 1/2$. □

3. *Proof.* Let $f(x) := (1+\sin(nx))(1+x/n)^n\,e^{-2x}$. Then we can easily verify that

$$0 \leq f(x) \leq 2\,e^{-x}.$$

Hence the function $g(b) := \int_0^b f(x)\,dx$ is an *increasing* and *bounded* function as $b \to \infty$. Therefore the limit

$$\lim_{b\to\infty} g(b) = \lim_{b\to\infty} \int_0^b f(x)\,dx$$

exists. □

4. (a) It is true. See Problem 7.2.2 on the Ash's book, page 125.
   (b) It is false. See Problem 7.2.3 on the Ash's book, page 125.

5. *Proof.* If $m(E_k) = \infty$ for some k, then $m(E_n) = \infty$ for all $n \geq k$ so that

$$m\Big(\bigcup_{n=1}^{\infty} E_n\Big) \geq m(E_k) = \infty = \lim_{n\to\infty} m(E_n).$$

Now we assume $m(E_n) < \infty$ for every n. Let $E_0 := \emptyset$ and $F_n := E_n - E_{n-1}$ for $n = 1, 2, \cdots$. Then $\{F_n\}$ is a collection of pairwise disjoint, measurable sets. Therefore we have

$$\begin{aligned} m\Big(\bigcup_{n=1}^{\infty} E_n\Big) &= m\Big(\bigcup_{n=1}^{\infty} F_n\Big) \\ &= \sum_{n=1}^{\infty} m(F_n) \quad \text{by countable additivity} \\ &= \sum_{n=1}^{\infty} \big(m(E_n) - m(E_{n-1})\big) \quad \text{for } E_n \supset E_{n-1} \text{ and } m(E_n) < \infty \\ &= \lim_{n\to\infty} m(E_n). \end{aligned}$$

□

6. Let $a_n := n(e^{-1/n} - 1)$. Then it holds that

$$\lim_{n\to\infty} a_n = -\lim_{n\to\infty} \frac{e^0 - e^{0-1/n}}{1/n} = -\frac{d}{dx}e^x\Big|_{x=0} = -1,$$

whence $\{a_n\}$ is a bounded sequence. Therefore, by the Lebesgue Dominated Convergence theorem, we have

$$\begin{aligned} \int_0^n n(e^{-x-1/n} - e^{-x})\,dx &= \int_0^n a_n e^{-x}\,dx = \int_{\mathbf{R}} \chi_{[0,n]} a_n e^{-x}\,dm \\ &\to \int_{[0,\infty]} (-e^{-x})\,dm = -\int_0^{\infty} e^{-x}\,dx = -1. \end{aligned}$$